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A Theory of Superstructures

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1 Axiom

We shall consider a first order theory of a language $\mathcal{L}_{\in} = \{\in\}$ on the classical logic with equality "$=$", where the symbol $\in$ is a membership relation.

We adopt the following abbreviations:

$$
\begin{aligned}
\mathrm{Set}(x) &\equiv \exists y\, y \in x,\\
\forall x \in y\ \varphi(x) &\equiv \forall x\ [x \in y \Rightarrow \varphi(x)],\\
\exists x \in y\ \varphi(x) &\equiv \exists x\ [x \in y \wedge \varphi(x)],\\
x \subseteq y &\equiv \forall z \in x\ z \in y,\\
x \notin y &\equiv \neg\, x \in y,\\
x \subsetneq y &\equiv x \subseteq y \wedge \exists u \in y\ u \notin x,\\
\mathrm{Trans}(x) &\equiv \forall y \in x\ y \subseteq x,\\
\forall x \subseteq y\ \varphi(x) &\equiv \forall x\ [x \subseteq y \Rightarrow \varphi(x)],\\
\mathrm{Wo}_{\subseteq}(x) &\equiv \forall y \subseteq x\ [\mathrm{Set}(y) \Rightarrow \exists u \in y\ \forall z \in y\ u \subseteq z] \wedge \forall z \in x\ \ \mathrm{Set}(z),\\
\mathrm{Mater}(x, y) &\equiv \forall z \in x\ \exists u \in y\ z \in u,\\
\exists! x\ \varphi(x) &\equiv \exists x\ \varphi(x) \wedge \forall x_1\ \forall x_2\ [\varphi(x_1) \wedge \varphi(x_2) \Rightarrow x_1 = x_2],\\
\exists! x \in y\ \varphi(x) &\equiv \exists! x\ [x \in y \wedge \varphi(x)].
\end{aligned}
$$

We call $\mathrm{Mater}(x, y)$ that x is a set of materials of y. In **ZF** set theory, $\mathrm{Mater}(x, y)$ means that x is a subset of union of y.

A formula φ of $\mathcal{L}_{\in}$ is *restricted* or *bounded* if all quantifiers in φ are of either form $\forall x \in y$ or $\exists x \in y$.

Here is an axiom system of a theory of Superstructures.

1. Extensionality of nonempty sets:

$$\forall x\ \forall y\ [\mathrm{Set}(x) \wedge x \subseteq y \wedge y \subseteq x \Rightarrow x = y].$$

2. Pair:

$$\forall x\, \forall y\, \exists u\, [x \in u \wedge y \in u].$$

3. Transitive superset:

$$\forall x\, \exists u\, [x \subseteq u \wedge \mathrm{Trans}(u)].$$

4. Power:

$$\forall x\, \exists u\, \forall y \subseteq x\; y \in u.$$

5. Infinity:

$$\exists u\, \left[\mathrm{Set}(u) \wedge \mathrm{Wo}_{\subseteq}(u) \wedge \forall y \in u\, \exists v \in u\; y \subsetneq v\right].$$

6. Strong foundation:

$$\forall x\, [\mathrm{Set}(x) \wedge \forall y \in x\, \exists u \in x\;\; \mathrm{Mater}(u, y) \Rightarrow \exists u \in x\, \neg\, \mathrm{Set}(u)].$$

7. Choice:

$$\forall x\, [\forall y \in x\, \exists u \in y\, \exists! v \in x\; u \in v \Rightarrow \exists w\, \forall y \in x\, \exists! u \in y\; u \in w].$$

8. Restricted separation: If $\varphi(y, z)$ is a restricted formula, then

$$\forall p\, \forall x\, \exists u\, \forall y\, [y \in u \Leftrightarrow y \in x \wedge \varphi(y, p)].$$

9. $\in$-induction schema:

$$\forall x\, [\forall y \in x\; \psi(y) \Rightarrow \psi(x)] \Rightarrow \forall x\; \psi(x).$$

We denote 1–9 by **SS** and 1–8 by $\mathbf{SS}_0$.

2 Universe

In this section, we consider the universe of $\mathbf{SS}_0$, and comulative hierarchy of **SS**.

By Infinity and Restricted separation, there is an a such that $\neg\,\mathrm{Set}(a)$, and by Power, there is b such that

$$\forall x \subseteq a\; x \in b, \text{ or } \forall x\, [\neg\,\mathrm{Set}(x) \Rightarrow x \in b].$$

By Restricted separation and Extensionality, there is a unique $\blacktriangledown$ such that

$$\forall x\ [x \in \blacktriangledown \Leftrightarrow \neg\,\mathrm{Set}(x)].$$

By Pair and Restricted separation, there is an unordered pair c for every a and b such that

$$\forall x\big[x \in c \Leftrightarrow [c = a \vee c = b]\big].$$

We denote such c by $\{a, b\}$ and $\{a, a\}$ by $\{a\}$. We define an ordered pair $\langle a, b\rangle$ by $\big\{\{a\}, \{a, b\}\big\}$.

Let $\varphi(x)$ be a restricted formula and suppose $\exists x \in a\ \varphi(x)$. Then, by Restricted separation and Extensionality of nonempty sets, there is a unique b such that

$$\forall x\ [x \in b \Leftrightarrow x \in a \wedge \varphi(x)].$$

We denote such b by $\{x \in a \mid \varphi(x)\}$.

By Power, there is a b for each a

$$\forall x \subseteq a\ x \in b.$$

We denote $\{x \in b \mid x \subseteq a\}$ by $\mathcal{P}_{\blacktriangledown}(a)$. Note that $\blacktriangledown \subseteq \mathcal{P}_{\blacktriangledown}(a)$ for every a.

By Transitive superset, for every x, there is t such that $\mathrm{Trans}(t) \wedge x \subseteq t$, define a transitive closure of x by:

$$\mathrm{TC}(x) = \begin{cases} x & \text{if } x \in \blacktriangledown \\ \big\{y \in t \bigm| \forall z \in \mathcal{P}_{\blacktriangledown}(t)\ [\mathrm{Trans}(z) \wedge x \subseteq z \Rightarrow y \in z]\big\} & \text{if } x \notin \blacktriangledown \end{cases}.$$

When $a \not\subseteq \blacktriangledown$, we denote the union $\big\{x \in \mathrm{TC}(a) \bigm| \exists y \in a\ x \in y\big\}$ by $\bigcup a$. When $\{a, b\} \not\subseteq \blacktriangledown$, we denote $\bigcup\{a, b\}$ by $a \cup b$.

As in **ZF**, we define maps, injections, surjections, bijections.

By Infinity, fixing α such that

$$\alpha \notin \blacktriangledown \wedge \mathrm{Wo}_{\subseteq}(\alpha) \wedge \forall y \in \alpha\ \exists v \in \alpha\ y \subsetneq v,$$

there is a unique $\subseteq$-least element 0_α in α: $\forall x \in \alpha\ 0_\alpha \subseteq x$. For every $x \in a$, there is unique x' such that

$$\forall y \in \alpha\ [x' \subseteq y \Leftrightarrow x \subsetneq y].$$

We denote such x' by $x +_\alpha 1$. We can define a minimal unbounded well-ordered set $\mathbb{N}_\alpha$ with order relation $\subseteq$ by

$$\mathbb{N}_\alpha = \Big\{x \in \alpha \Bigm| \forall y \in \alpha\ \big[0 \subsetneq y \wedge \forall z \in \alpha[z \subsetneq y \Rightarrow z +_\alpha 1 \subsetneq y] \Rightarrow x \subsetneq y\big]\Big\}.$$

Then we have Restricted induction principle:

$$\varphi(0) \wedge \forall n \in \mathbb{N}_\alpha \left[\varphi(n) \Rightarrow \varphi(n +_\alpha 1)\right] \Rightarrow \forall n \in \mathbb{N}_\alpha\, \varphi(n),$$

where $\varphi(n)$ is restricted. Then we have that $\mathbb{N}_\alpha$ is unique up to isomorphism, so we denote a structure of natural numbers by $\langle \mathbb{N}, \leq, +1, 0\rangle$.

Since $u \in y$ implies $\mathrm{Mater}(y, u)$, we have, by Strong foundation, foundation principle:

$$\forall x\, [\mathrm{Set}(x) \Rightarrow \exists y \in x\, \forall u \in x\; u \notin y].$$

We shall show dual foundation principle:

$$\forall x\, [\mathrm{Set}(x) \Rightarrow \exists y \in x\, \forall u \in x\; y \notin u].$$

Suppose, on contrary, there is x such that $\mathrm{Set}(x)$ and $\forall y \in x\; \exists u \in x\; y \in u$. Since $\mathrm{Mater}\big(\mathrm{TC}(x), \mathrm{TC}(x)\big)$, we have, by Strong foundation, there is $a \in$ [illegible] such that $a \in \big\{\mathrm{TC}(x)\big\}$, which is contradiction.

Let $\mathbb{N}$ be a structure of natural numbers. we define the predicate "x has rank n" by

$$\begin{aligned}
\rho(n, x) &\equiv \bar{\rho}\big(n, \mathrm{TC}(x) \cup \{x\}, x\big),\\
\bar{\rho}(n, t, x) &\equiv \exists f \colon t \to \mathbb{N} \Big[\forall y \in t\; f(y) = \bigcup \big\{k \in \mathbb{N} \mid k = 0 \vee \exists z \in y\; k = f(z) + 1\big\}\\
&\qquad\qquad \wedge n = f(x)\Big].
\end{aligned}$$

Then every x has a unique rank.

In **SS**, applying the following $\psi(x)$ to $\in$-induction schema, we have obtained comulative hierarchy W_n. We cannot prove that there is W_n for every $n \in \mathbb{N}$.

$$\psi(x) \equiv \forall n \in \mathbb{N}\big[\rho(n, x) \Rightarrow \exists W_n\, \forall y\, [y \in W_n \Leftrightarrow \exists k \leq n\; \rho(k, y)]\big].$$

3 Models

We construct models for **SS** in **ZFC**. We say a model W is ***ZF**-standard* if the membership relation $\in$ of W is that of **ZFC**.

Given a set X, we define the iterated power set $V_n(X)$ over X recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set X is said to be a *base set* if $\varnothing \notin X$ and each element of X is disjoint from $V(X)$.

If X is a base set then $V(X)$ is a **ZF**-standard model for **SS**. In $V(X)$, we see $X \cup \{\varnothing\} = \blacktriangledown$ and $\mathcal{P}_{\blacktriangledown}(a) = \mathcal{P}(a) \cup \blacktriangledown$.

Let X and Y are infinite base sets, and let $j\colon V(X) \to V(Y)$ be a nontrivial bounded elementary embedding — $\langle V(X), V(Y), j\rangle$ is a nonstandard universe. Then the transitive closure W of $\operatorname{ran} j$ within $V(Y)$ is a model for **SS**. In W, we see $j(\mathbb{N})$ is a structure of natural numbers if $\mathbb{N}$ is a structure of natural numbers in $V(X)$, and there is no W_ν for nonstandard $\nu \in j(\mathbb{N}) \setminus j``\mathbb{N}$.